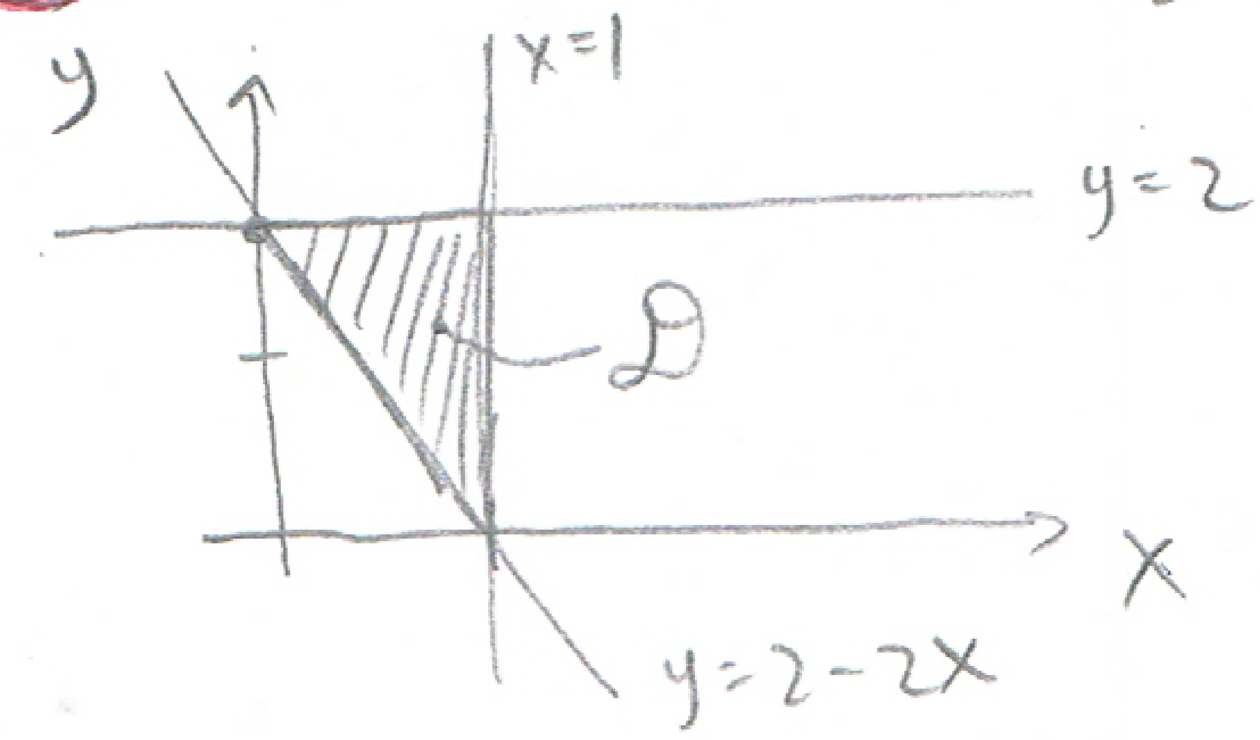
Math 2E Spring 2016 MT Practice

1 Note $2x+y=2 \Rightarrow y=2-2x$.



(a) $$\int_{x=0}^{1}\int_{y=2-2x}^{2} e^{x-y}\,dy\,dx$$

and $$\int_{y=0}^{2}\int_{x=\frac{2-y}{2}}^{1} e^{x-y}\,dy\,dx$$

$e^{x-y} = e^{x}\cdot e^{-y}$

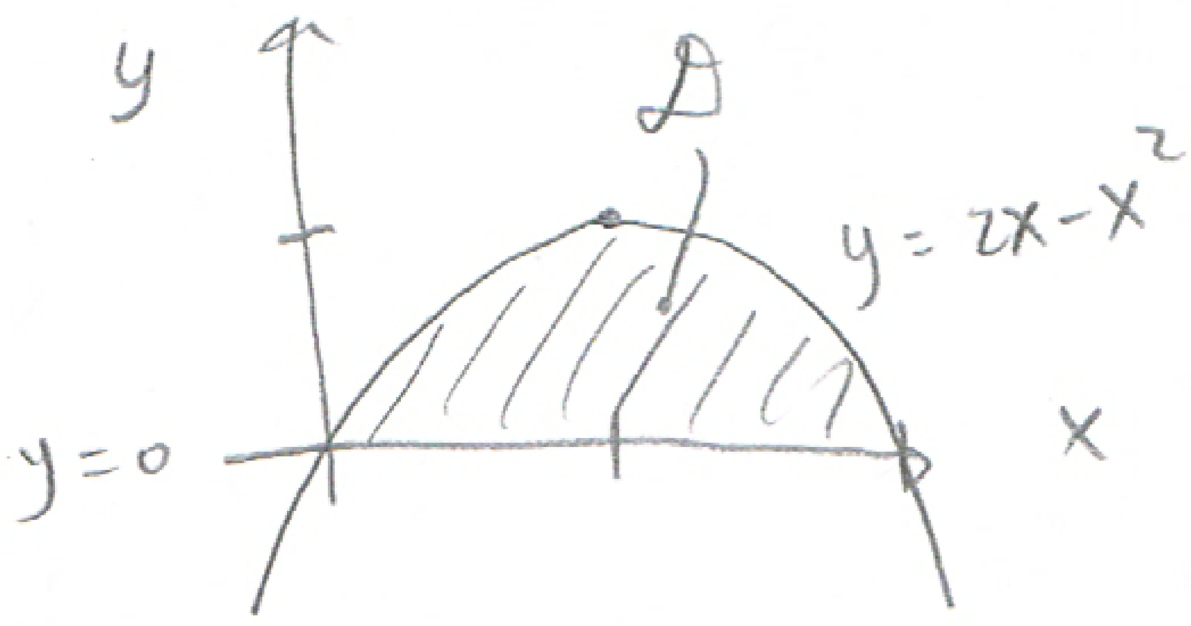
(b) $$\int_{x=0}^{1}\int_{y=2-2x}^{2} e^{x-y}\,dy\,dx = \int_{x=0}^{1} -e^{x-y}\Big|_{y=2-2x}^{2}\,dx$$

$$= \int_{x=0}^{1}\left(-e^{x-2}+e^{3x-2}\right)dx = -e^{x-2}+\frac{e^{3x-2}}{3}\Big|_{0}^{1}$$

$$= -e^{-1}+\frac{e^{1}}{3}+e^{-2}-\frac{e^{-2}}{3} = \boxed{\frac{e}{3}-\frac{1}{e}+\frac{2}{3e^{2}}}$$

2 Note $y=2x-x^2=-(x-1)^2+1$, vertex at $(1,1)$. $\Rightarrow$ $y-1=-(x-1)^2$,

$x=1\pm\sqrt{1-y}$



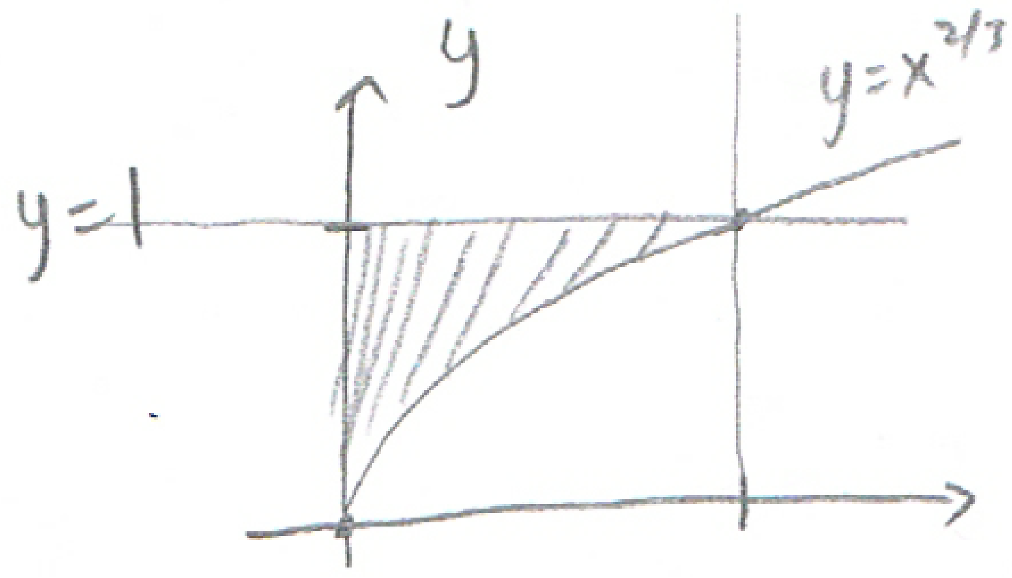
(a) $$\int_{x=0}^{2}\int_{y=0}^{2x-x^2} xy\,dy\,dx$$

and $$\int_{y=0}^{1}\int_{x=1-\sqrt{1-y}}^{1+\sqrt{1-y}} xy\,dx\,dy$$

(b) $$\int_{x=0}^{2}\int_{y=0}^{2x-x^2} xy\,dy\,dx = \int_0^2 x\,\frac{y^2}{2}\Big|_0^{2x-x^2}dx = \int_0^2 x\,\frac{(4x^2-4x^3+x^4)}{2}\,dx$$

$$= \frac{1}{2}\left(\frac{x^6}{6}-\frac{4x^5}{5}+x^4\right)\Big|_0^2 = \boxed{\left(\frac{64}{6}-\frac{128}{5}+16\right)\frac{1}{2}}$$

3 (a) $$\int_{x=0}^{1}\int_{y=x^{2/3}}^{1} x\cos(y^4)\,dy\,dx = \int_{y=0}^{1}\int_{x=0}^{y^{3/2}} x\cos(y^4)\,dx\,dy$$



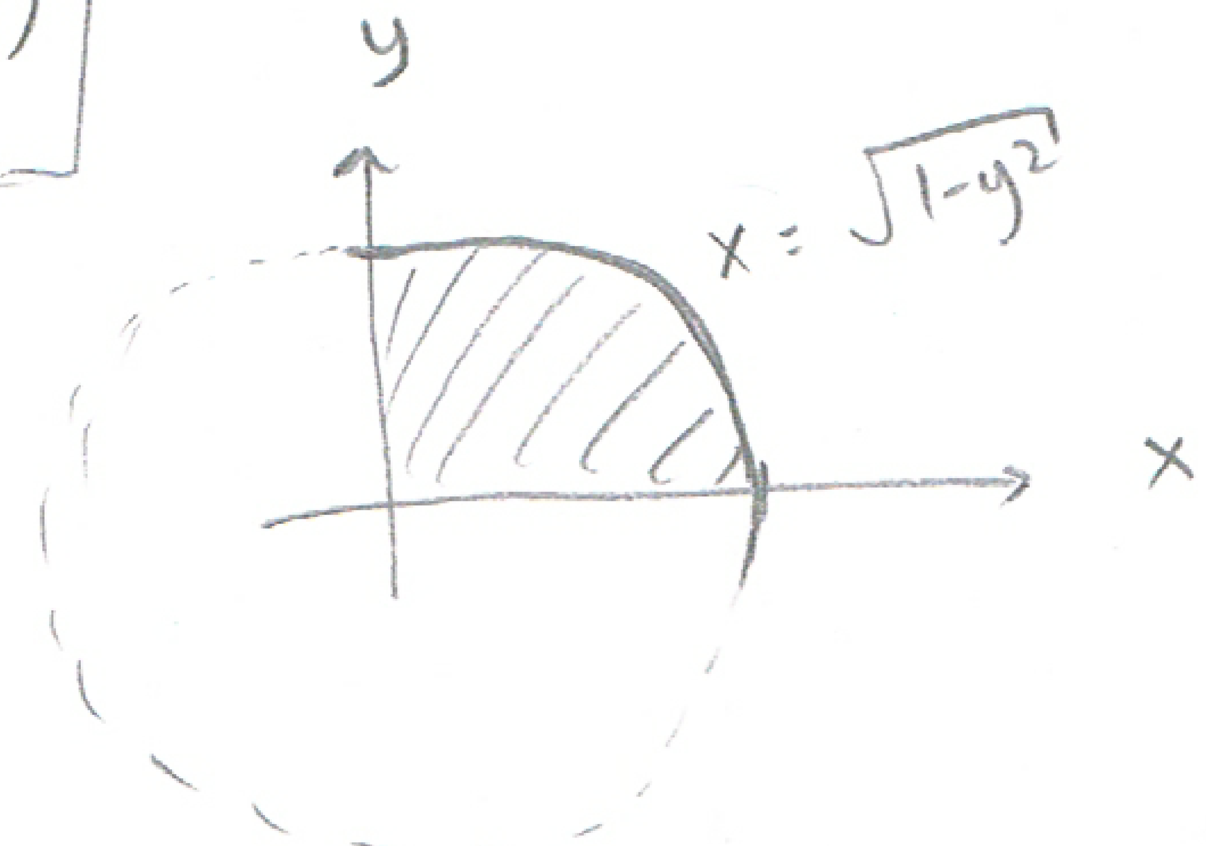
$y = x^{2/3} \Rightarrow x = y^{3/2}$

$$= \int_{y=0}^{1} \frac{x^2}{2}\cos(y^4)\Big|_{x=0}^{y^{3/2}}\,dy$$

$$= \int_0^1 \frac{y^3}{2}\cos(y^4)\,dy \overset{\substack{u=y^4\\ du=4y^3dy}}{=} \int_{u=0}^{u=1} \cos(u)\cdot\frac{du}{8}$$

$$= \frac{1}{8}\left(\sin(u)\Big|_0^1\right) = \boxed{\frac{1}{8}\sin(1)}$$

(b) $$\int_{y=0}^{1}\int_{x=0}^{\sqrt{1-y^2}} (x^2+y^2)^{2016}\,dx\,dy$$



polar!

$$= \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{1} (r^2)^{2016}\cdot r\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 r^{4032+1}\,dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{r^{4034}}{4034}\Big|_{r=0}^{1}\,d\theta = \boxed{\frac{\pi}{2}\cdot\frac{1}{4034}}$$

4 1^{st}, with the substitution, $\underline{x^2+xy+y^2}$ becomes: $\left(\begin{array}{l}\text{Plug in}\\ x=u+v\sqrt{3}\\ y=u-v\sqrt{3}\end{array}\right)$

$$(u+v\sqrt{3})^2 + (u+v\sqrt{3})(u-v\sqrt{3}) + (u-v\sqrt{3})^2 = (u^2+\cancel{2uv\sqrt{3}}+3v^2)+(u^2-3v^2) + (u^2-\cancel{2uv\sqrt{3}}+3v^2)$$

$$= \underline{3u^2+3v^2}$$

Thus, $x^2+xy+y^2=1 \Rightarrow u^2+v^2=\frac{1}{3}$

Thus, with $\text{Area} = \iint_{A_{\text{enclosed}\,xy}} 1 \, dA_{xy} \overset{\text{Chg. Vars}}{=} \iint_{\substack{\text{New}\\ A_{uv}}} 1 \cdot \underline{\underline{\boxed{J}\, dA_{uv}}}$,

We just need the uv-integral $A_{uv} \Rightarrow$ disk! $\boxed{u^2+v^2 \le \frac{1}{3}}$:)

For J, $J = \left|\det\begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}\right| = \left|\det\begin{bmatrix} 1 & \sqrt{3} \\ 1 & -\sqrt{3} \end{bmatrix}\right| = \boxed{|-2\sqrt{3}|} = \boxed{2\sqrt{3}}$

Thus, $\text{Area} = \iint_{A_{uv}} |2\sqrt{3}| \, dA_{uv} \overset{\text{polar}}{=} \int_{\theta=0}^{2\pi} \int_{r=0}^{\frac{1}{\sqrt{3}}} 2\sqrt{3} \; r\,dr\,d\theta$

$$= \cancel{2}\sqrt{3} \int_0^{2\pi} \frac{r^2}{\cancel{2}} \Big|_0^{\frac{1}{\sqrt{3}}} d\theta = \boxed{2\pi\sqrt{3} \cdot \frac{1}{3\cancel{\times}}}$$ is the area (with units^2)

5 $\int_{z=1}^{2} \int_{y=0}^{z} \int_{x=y}^{z} xyz \, dx\,dy\,dz = \int_{z=1}^{2} \int_{y=0}^{z} \frac{x^2}{2} yz \Big|_{x=y}^{x=z} dy\,dz$

$$= \int_{z=1}^{2} \int_{y=0}^{2} \left(\frac{yz^3}{2} - \frac{y^3 z}{2} \right) dy\,dz = \int_{z=1}^{2} \frac{y^2 z^3}{4} - \frac{y^4 z}{8} \Big|_0^{z} dz$$

$$= \int_{z=1}^{2} \left(\frac{z^5}{4} - \frac{z^5}{8} \right) dz = \int_{z=1}^{2} \frac{z^5}{8} \, dz$$

$$= \frac{z^6}{48} \Big|_{z=1}^{2} = \boxed{\frac{2^6 - 1}{48}}$$

6. First, $x'(t) = e^t\,dt$, $y'(t) = -e^{-t}\,dt$

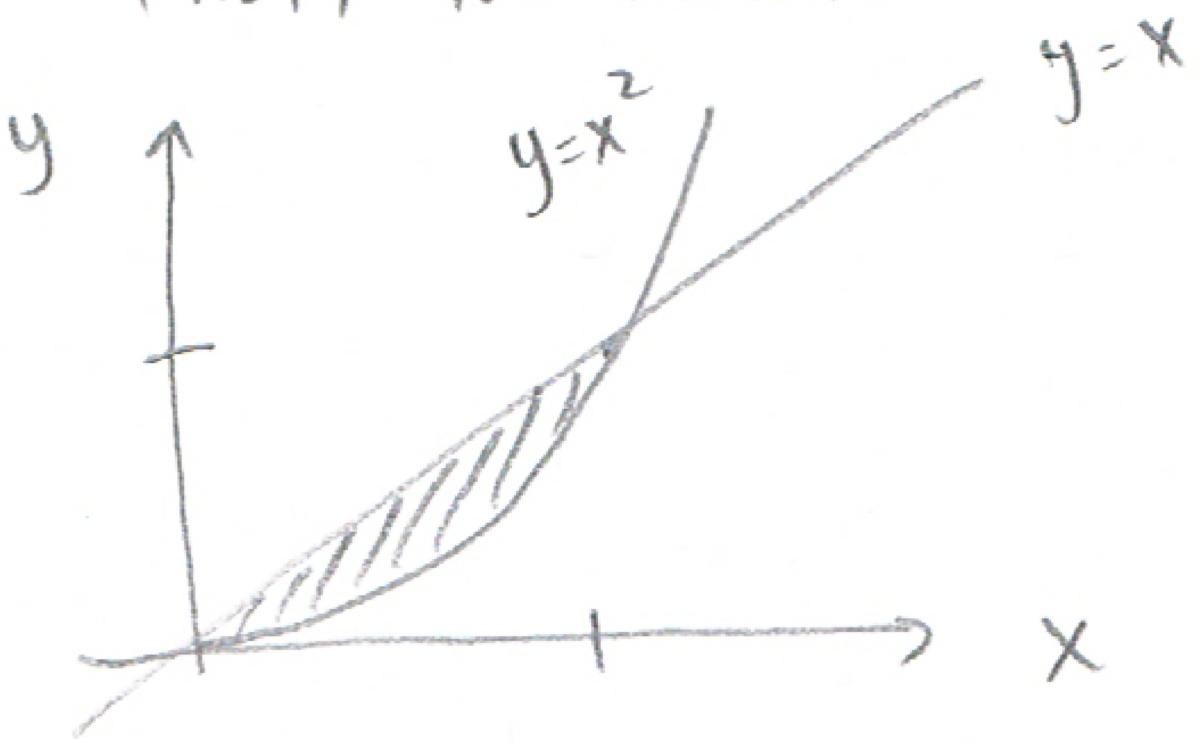
Thus, $\displaystyle\int_C xy\,dx + \ln(x)\,dy = \int_0^1 \cancel{e^t}\cdot\cancel{e^{-t}}\cdot(e^t\,dt) + \ln(e^t)\cdot(-e^{-t}\,d$

$$= \int_0^1 e^t\,dt - te^{-t}\,dt = e^t\Big|_0^1 + \int_0^1 \ominus te^{-t}\,dt \qquad \begin{array}{c|c} u=t & dv=-e^{-t} \\ \hline du=dt & v=\oplus e^{-t} \end{array}$$

$$= (e^1-1) + te^{-t}\Big|_0^1 - \int_0^1 e^{-t}\,dt$$

$$= (e-1) + (e^{-1}) + e^{-t}\Big|_0^1 = \boxed{e + 2e^{-1} - 2}$$

7. First, the domain:



Changing Order of Integral:

$$= \int_{y=0}^{1}\int_{x=y}^{x=\sqrt{y}} x\,f(y)\,dx\,dy$$

$$= \int_{y=0}^{1} \frac{x^2}{2} f(y)\Big|_{x=y}^{\sqrt{y}}\,dy$$

$$= \frac{1}{2}\int_{y=0}^{1} (y-y^2)\,f(y)\,dy \quad ✓$$

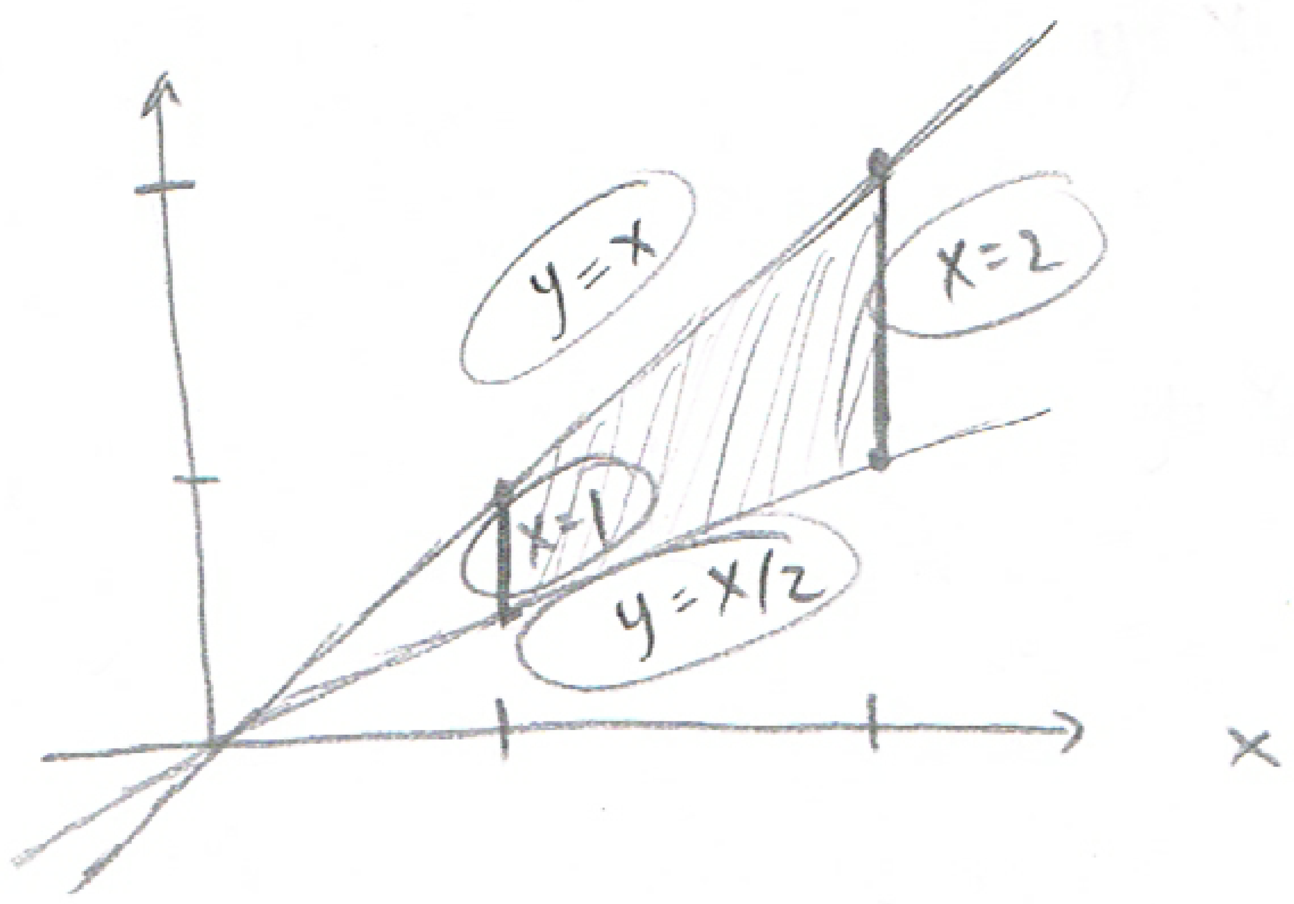
8. First, $\vec{F}(\vec{r}(t)) = \langle \sin^2 t\,(1+\sin 2t)^5,\ \frac{5}{3}\sin^3 t\,(1+\sin 2t)^4 \rangle$

Next, $\vec{r}\,'(t) = \langle \cos t,\ 2\cos 2t \rangle$.

Since $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$, $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Big(\sin^2 t\,(1+\sin 2t)^5 \cos t + \frac{5}{3}\sin^3 t\,(1+\sin 2t)\cdot 2\cos t\Big)\,dt$

★ We can re-visit in Ch. 16.3 ☺

(9) (i) Change to uv-domain First, the xy domain is:

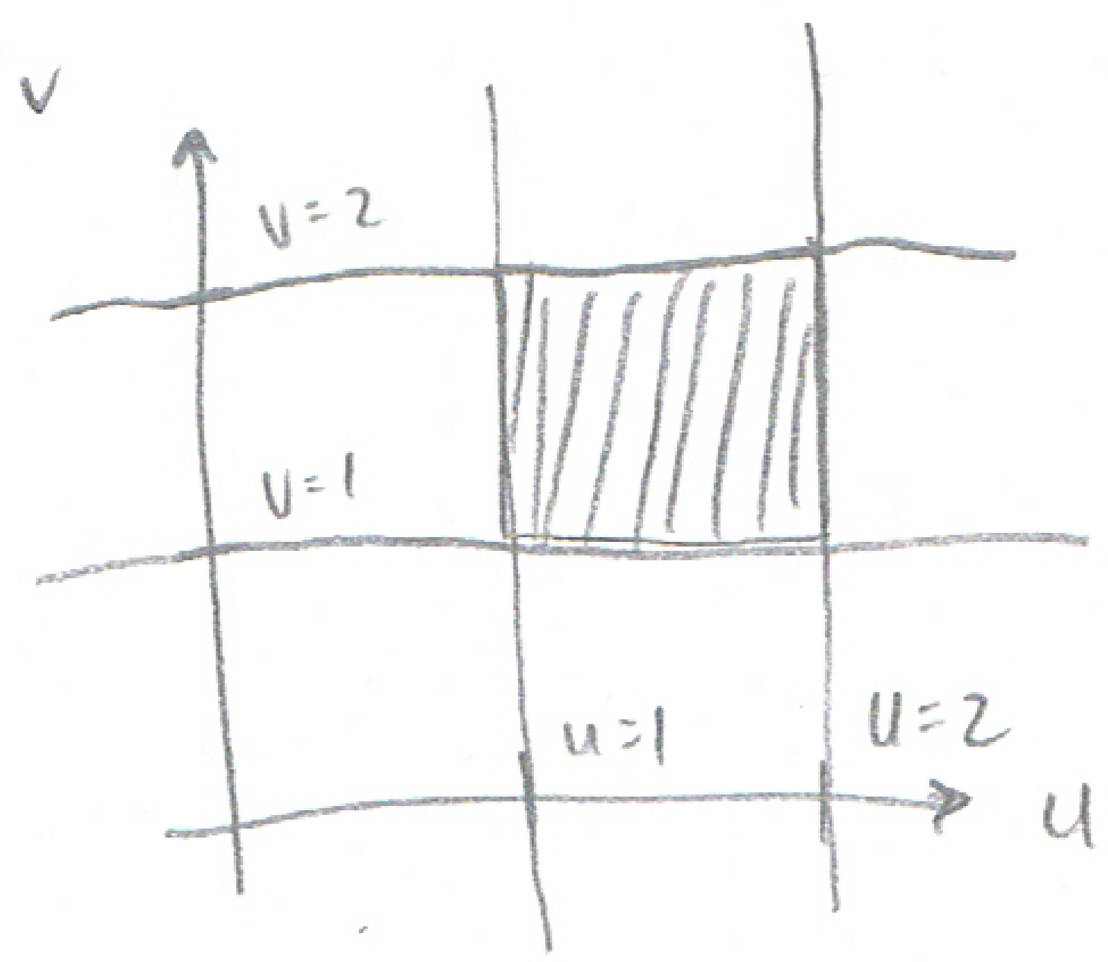


☆ There 4 boundary curves, $x = 1, 2$ and $y = x, \frac{x}{2}$.

We need to see how they change.

- $x = 1 \Rightarrow \boxed{u=1}$, and $y = \frac{u}{v} = \frac{1}{v}$ means (from $\frac{1}{2} \le y \le 1$) $\rightarrow$ $1 \le v \le 2$.
- $x = 2 \Rightarrow \boxed{u=2}$ and $y = \frac{u}{v} = \frac{2}{v}$ gives (from $1 \le y \le 2$) $\rightarrow$ $1 \le v \le 2$ (still)
- $y = x \Rightarrow \frac{u}{v} = y = x = u$; $\frac{u}{v} = u$, so $\boxed{v=1}$
- $y = \frac{x}{2} \Rightarrow \frac{u}{v} = y = \frac{x}{2} = \frac{u}{2}$; $\frac{u}{v} = \frac{u}{2}$, so $\boxed{v=2}$

uv domain: Box!



(ii) Jacobian $J = \left| \det \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} \right| = \left| \det \begin{bmatrix} 1 & 0 \\ \frac{1}{v} & -\frac{u}{v^2} \end{bmatrix} \right| = \left| -\frac{u}{v^2} - 0 \right|$

$$\boxed{J = \frac{u}{v^2}}$$

(iii) Integral in uv, $= \int_{u=1}^{2} \int_{v=1}^{2} \frac{u}{\left(\frac{u}{v}\right)^2} \sin\left(\frac{\pi u}{\frac{u}{v}}\right) \cdot \frac{u}{v^2} \, dv\, du$

$$= \int_{u=1}^{2} \int_{v=1}^{2} \frac{\cancel{v^2}}{\cancel{u}} \sin(\pi v) \cdot \frac{\cancel{u}}{\cancel{v^2}} \, dv\, du = \int_{u=1}^{2} -\frac{\cos \pi v}{\pi} \Big|_{v=1}^{2} du$$

No v-dep

$$= \int_{u=1}^{2} \frac{1}{\pi}(-1-1) = -\frac{2}{\pi} \cdot (2-1) = \boxed{-\frac{2}{\pi}}$$

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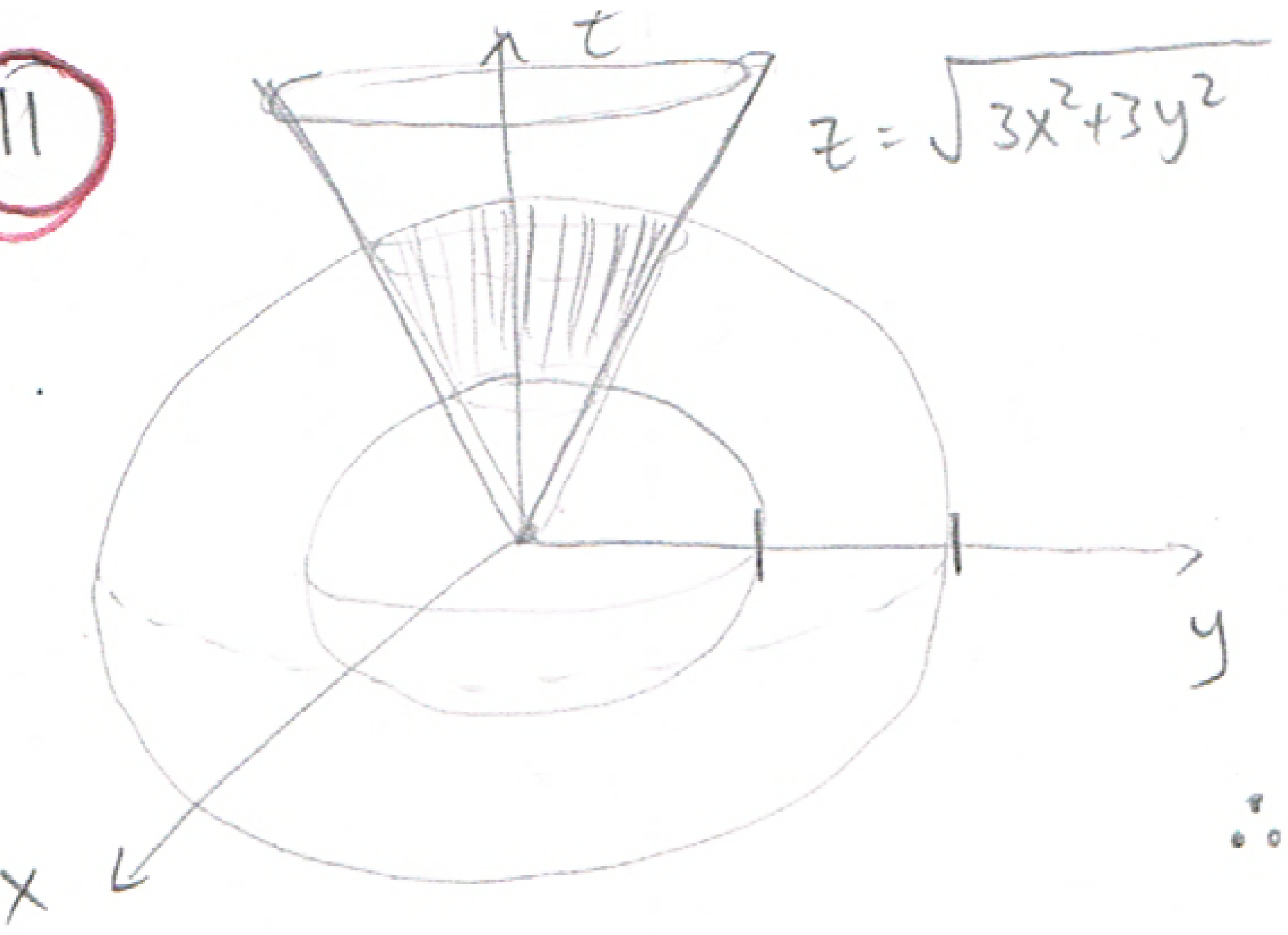
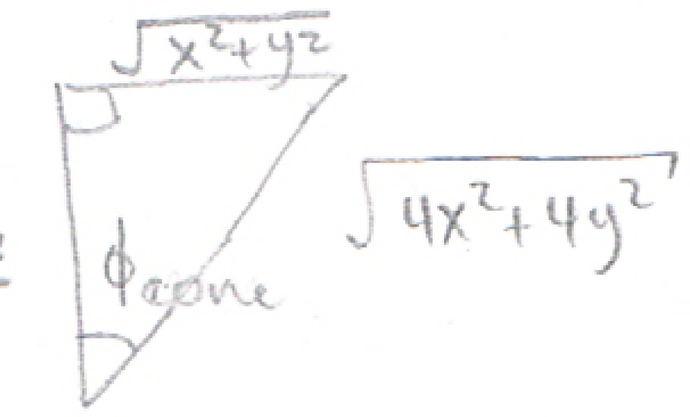


- We go all around in θ
- We go only until the cone in ϕ
- We're between spheres $\rho = 1, 2$.

∴ When we write $\iiint_E \sqrt{x^2+y^2+z^2}\,dV$ in spherical,

$0 \le \theta \le 2\pi$, $1 \le \rho \le 2$, and $0 \le \phi \le$ Angle from cone.

For cone's angle ϕ_{cone} on a cross-section, it looks like



so $\phi_{cone} = \tan^{-1}\left(\frac{z}{\sqrt{x^2+y^2}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$ ✓

or, $= \cos^{-1}\left(\frac{z}{\rho}\right) = \cos^{-1}\left(\frac{\sqrt{3}\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+3x^2+3y^2}}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$ ✓

so $0 \le \phi \le \pi/6$. Then, the integral is

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/6}\int_{\rho=1}^{2} \rho\cdot\left(\rho^2\sin\phi\,d\rho\,d\phi\,d\theta\right) = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\frac{\pi}{6}} \frac{\rho^4}{4}\sin\phi\Big|_{\rho=1}^{2} d\phi\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/6}\frac{15}{4}\sin\phi\,d\phi\,d\theta = \int_{\theta=0}^{2\pi}\frac{15}{4}(-\cos\phi)\Big|_{0}^{\pi/6} d\theta$$

θ-indep

$$= \boxed{2\pi\cdot\frac{15}{4}\cdot\left(1-\frac{\sqrt{3}}{2}\right)}$$

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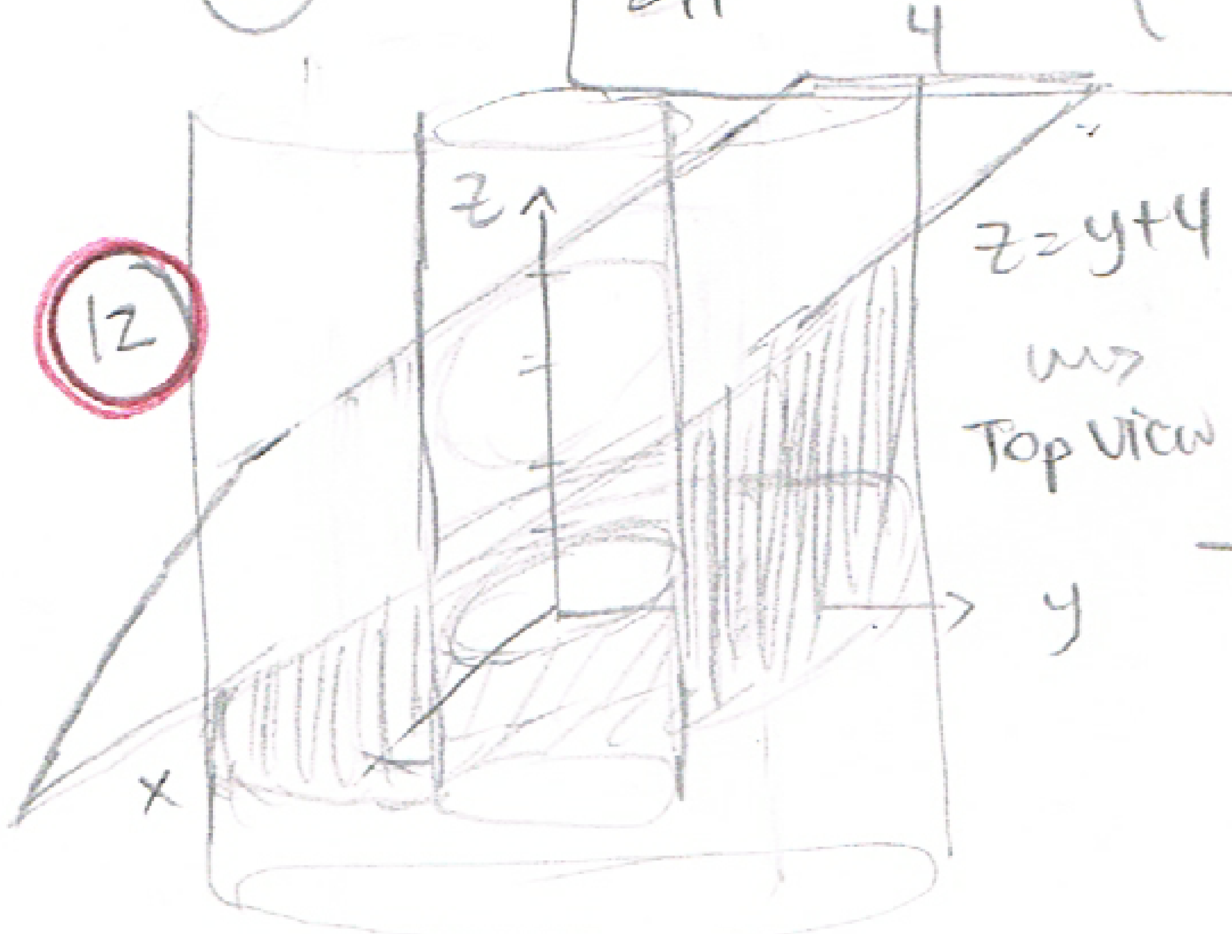
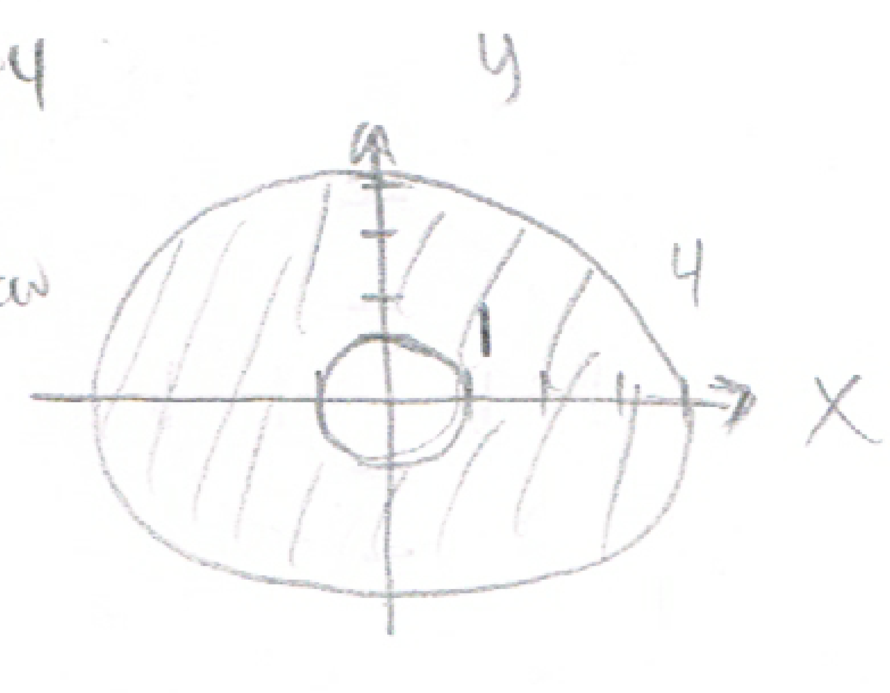




So we should use cylindrical, $0 \le \theta \le 2\pi$, $1 \le r \le 4$, and z is between the xy plane, $z=0$, and $z=y+4$.

Cylindrical ⇒

$$\int_{\theta=0}^{2\pi}\int_{r=1}^{4}\int_{z=0}^{4+r\sin\theta}(r\cos\theta - r\sin\theta)\cdot r\,dz\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=1}^{4}\int_{z=0}^{4+r\sin\theta} r^2(\cos\theta-\sin\theta)\,dz\,dr\,d\theta.$$

★ No z-dependence in the integral!

$$= \int_{\theta=0}^{2\pi}\int_{r=1}^{4} r^2(\cos\theta - \sin\theta)\cdot(4 + r\sin\theta)\, dr\, d\theta$$

Fubini

$$= \int_{r=1}^{4}\int_{\theta=0}^{2\pi} r^2\left[4\cos\theta - 4\sin\theta + \underbrace{r\sin\theta\cos\theta}_{\frac{1}{2}r\sin 2\theta} - \underbrace{r\sin^2\theta}_{\frac{r}{2}(1-\cos 2\theta)}\right] d\theta\, dr$$

☆ Four of these integrals vanish by periodicity!

periodic

$$= \int_{r=1}^{4}\int_{\theta=0}^{2\pi} -r^2\cdot\frac{r}{2}\, d\theta\, dr = \frac{2\pi}{2}\int_{r=1}^{4} -r^3\, dr$$

$$= -\frac{\pi r^4}{4}\Big|_{r=1}^{4} = \boxed{-\frac{255\pi}{4}} \quad (4^4 = 256).$$

13 (a) $L = \int_C 1\, ds.$ $x'(t) = -\cancel{\sin t} + \cancel{\sin t} + t\cos t = t\cos t$

$y'(t) = \cancel{\cos t} - \cancel{\cos t} + t\sin t = t\sin t$

$$L = \int_0^{10}\sqrt{t^2\cos^2 t + t^2\sin^2 t}\, dt = \int_0^{10} t\,\overbrace{\sqrt{\cos^2 + \sin^2}}^{1}\, dt = \frac{t^2}{2}\Big|_0^{10} = \boxed{50}$$

(b) $$\int_C x\, ds = \int_0^{10}(\cos t + t\sin t)\cdot t\, dt = \int_0^{10}(t\cos t + t^2\sin t)\, dt$$

parts

$$= t\sin t\Big|_0^{10} - \int_0^{10}\sin t \quad - t^2\cos t\Big|_0^{10} + \int_0^{10} 2t\cos t\, dt$$

we did this already

$$= 3\left[10\sin 10 + \cos t\Big|_0^{10}\right] - 100\cos 10$$

$$= \boxed{3\left[10\sin(10) + \cos(10) - 1\right] - 100\cos(10).}$$

(14)

All parts need curve C broken into 2 pieces

(a) C_1: $\vec{r}(t) = (1-t)\langle 1,0\rangle + t\langle 0,0\rangle = \langle 1-t, 0\rangle$, $0 \le t \le 1$.

☆ Note $y \equiv 0$ so $dy = 0$ $(y'(t) = 0) \Rightarrow \int_{C_1}$ gives nothing!

so $\int_C = \int_{C_2}$ only, where C_2 is the parabola.

C_2: Let $x = t$, $y = t^2$ then, and $0 \le t \le \pi$ hence, $\begin{pmatrix} x'(t) = 1 \\ y'(t) = 2t \end{pmatrix}$

$$\Rightarrow \int_{C_2} (x^2 y + \sin x)\, dy = \int_0^{\pi} (t^4 + \sin t)\cdot 2t\, dt = \frac{2t^6}{6}\Big|_0^{\pi} + \int_0^{\pi} 2t \sin t\, dt$$

By parts

$$\stackrel{\text{By parts}}{=} \frac{2\pi^6}{6} + 2\left[-t\cos t\Big|_0^{\pi} + \int_0^{\pi} \cos t\, dt\right] = \frac{\pi^6}{3} + 2\left[+\pi + \underbrace{\sin t\Big|_0^{\pi}}_{\text{zero}}\right]$$

$$= \boxed{\frac{\pi^6}{3} + 2\pi}$$

(b) C_1: Is now $\vec{r}(t) = (1-t)\langle 0,1\rangle + t\langle 0,0\rangle = \langle 0, 1-t\rangle$, $0 \le t \le 1$

Now $x \equiv 0$ on $C_1 \Rightarrow (x^2 y + \sin x) \equiv 0$ so again, $\int_{C_1} \to 0$.

So the only contribution is from C_2, we get same answer, $\boxed{2\pi + \frac{\pi^6}{3}}$

(c) C_1: Let us use $\int_{C_1} = \ominus \int_{\ominus C_1}$, so $\ominus C_1$ goes from $(0,0)$ to $(1,1)$

so $\ominus C_1$: Is $\vec{r}(t) = (1-t)\langle 0,0\rangle + t\langle 1,1\rangle = \langle t, t\rangle$, $0 \le t \le 1$.

Now, $$\int_{\ominus C_1} (x^2 y + \sin x)\, dy = \int_0^1 (t^3 + \sin t)\, 1\, dt$$

$$= \frac{t^4}{4} - \cos t\Big|_0^1 = \frac{1}{4} + 1 - \cos(1)$$

Thus, $$\int_{C_1} = \ominus \int_{\ominus C_1} = -\left[\frac{5}{4} - \cos(1)\right]$$

And $\int_C = \int_{C_1} + \int_{C_2} \Rightarrow$ $\boxed{\text{Answer} = 2\pi + \frac{\pi^6}{3} + \cos(1) - \frac{5}{4}}$

16 (a) If we plug in $x=pu$, $y=pv$, we should equivalently take R_q to S.

From R_q, as long as $p>0$, $x=pu\geq 0$, $y=pv\geq 0 \overset{\text{imply}}{\Rightarrow} u\geq 0, v\geq 0$ ✓

For $\sqrt{x}+\sqrt{y}\leq q$, we read $\sqrt{pu}+\sqrt{pv}\leq q$; $\sqrt{p}\left(\sqrt{u}+\sqrt{v}\right)\leq q$.

We will have $\sqrt{u}+\sqrt{v}\leq 1$ if $\sqrt{p}=q$, so $\boxed{p=q^2}$ is the answer.

(b) We just need J, $J=\det\begin{bmatrix}x_u & x_v\\ y_u & y_v\end{bmatrix}=\det\begin{bmatrix}p & 0\\ 0 & p\end{bmatrix}=p^2$.

Thus, $A(q)=\iint_S p^2\,du\,dv=\iint_S q^4\,du\,dv=q^4\cdot\text{Area}(S)$.

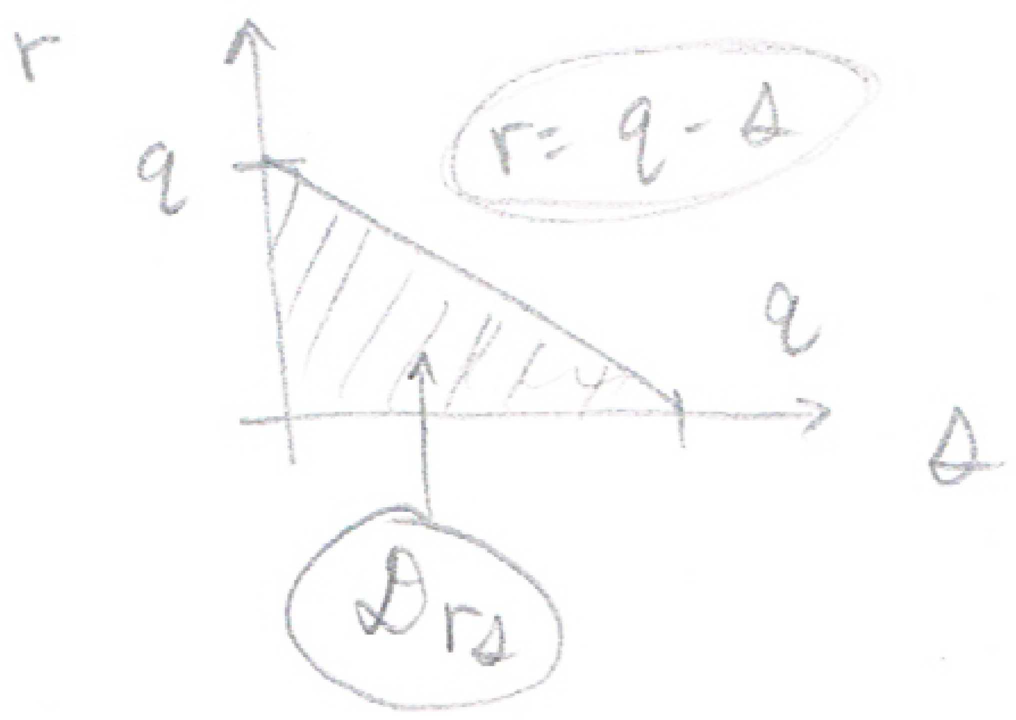
(c) If we double q, the area is 16 times, due to q^4,

In other words, $A(2q)=16\left(q^4\,\text{Area}(S)\right)=16A(q)$.

(d) With $x=r^2$, $y=s^2$, (i) R_q becomes: $x\geq 0 \Rightarrow r\geq 0$

$y\geq 0 \Rightarrow s\geq 0$.

Mainly, $\sqrt{x}+\sqrt{y}\leq q \rightsquigarrow r+s\leq q$



(ii) $J=\det\begin{vmatrix}x_r & x_s\\ y_r & y_s\end{vmatrix}=\det\begin{bmatrix}2r & 0\\ 0 & 2s\end{bmatrix}=4rs$

(iii) $A(q)=\iint_{R_q} 1\,dA=\iint_{D_{rs}} 1\,(J\,dr\,ds)$

$$=\int_{s=0}^{q}\int_{r=0}^{q-s}4rs\,dr\,ds=\int_{s=0}^{q}2r^2s\Big|_0^{q-s}ds$$

$$=\int_{s=0}^{q}2\left(q^2s-2qs^2+s^3\right)ds=q^2s^2-\frac{4qs^3}{3}+\frac{s^4}{2}\Big|_0^{q}$$

$$=q^4-\frac{4}{3}q^4+\frac{q^4}{2}=q^4\left(\frac{3}{2}-\frac{4}{3}\right)=\boxed{\frac{q^4}{6}}$$

(This is consistent with part (c))